

Computing Tree Decompositions with FlowCutter: PACE 2017 Submission*

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Abstract

We describe the algorithm behind our PACE 2017 submission to the heuristic tree decomposition computation track. It was the only competitor to solve all instances and won a tight second place. The algorithm was originally developed in the context of accelerating shortest path computation on road graphs using multilevel partitions. We illustrate how this seemingly unrelated field fits into tree decomposition and parameterized complexity theory.

1 Introduction

Tree decompositions are an established graph decomposition methodology. They can be used to measure how “close” a graph is to a tree. The width of a decomposition indicates how “tree-like” a graph is. We refer to [5], [6], and [7] for an overview of the field.

Tree decompositions are often used in combination with parameterized complexity. Many NP-hard problems can be solved in linear time on tree graphs. The corresponding linear time algorithms can often be generalized to tree decompositions. The generalized algorithms’ running times are usually linear in the graph size but super-polynomial in the width of the decomposition.

A lot of theoretical research with in-depth results into this field exists. However, it is unclear whether these results translate into algorithms that are efficient in practice. A core issue consists of finding decompositions of small size. Without an algorithm that is fast in practice and computes tree decompositions with a reasonably small width, no algorithm parameterized in the tree width is usable in practice.

To bridge this gap between theory and practice, the PACE implementation challenge was created in 2016 [9]. Track A2 focuses on computing tree decompositions of small width within a given amount of time. Because of its practical relevance, the challenge was repeated in 2017. In this paper, we describe the FlowCutter submission to the PACE 2017 Track A2 contest. It was the only submission to solve all test instances and won a close second place.

FlowCutter was originally developed to accelerate shortest path computations on road graphs. Fortunately, it is applicable in a significantly broader context. The corresponding PACE 2016 and 2017 submissions demonstrate this. For a survey of shortest path computation algorithms, we refer to [3]. Tree decompositions are rarely used explicitly in this community. However, very similar concepts are used.

Many shortest path acceleration techniques exist. They work in two phases: In the preprocessing phase, the graph is transformed. In the query phase, shortest paths are computed using the transformed graph. The preprocessing phase may be slow. The query phase should be fast. The motivation is that the preprocessing only needs to run when the map is updated. This is assumed to be rare. For example, in the preprocessing phase one can construct a small tree decomposition. In the query phase, the decomposition can be used to compute shortest paths.

These shortest path computation algorithms are an example of parameterized complexity. The algorithms assume that a small tree decomposition is given in the input. The complexity of the shortest path problem is bounded in terms of the tree width. The difference to regular fixed parameter tractability theory is that the considered problem is not NP-hard.

Two common shortest path techniques are Contraction Hierarchies (CH) [14] and Multilevel Dijkstra (MLD) [18, 17, 10]. None of the original publications mention tree decompositions. However, very similar concepts are used. The similarity is a comparatively recent discovery. We illustrate the relationship in this paper.

A question often asked is what algorithms big companies with mapping services use. Unfortunately, most companies do not publicly share this information. A welcome exception is Microsoft. They stated that the Bing routing service uses MLD [11]. This illustrates that research in this domain has direct practical implications.

The relation between multilevel graph partitions and tree decompositions is not well known as none of the cited survey articles mention it [5, 6, 7, 3]. The relationship was hinted in [1]. In this paper, we make the connection clearer and

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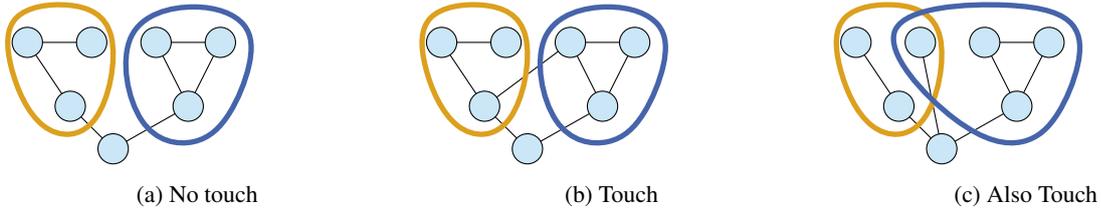


Figure 1: Graph with two cells illustrating the touch definition.

more explicit. There is a one-to-one correspondence between multilevel graph partitions and rooted tree decompositions. Using this one-to-one relationship, we can also establish a link between CH and MLD via tree decomposition theory. We believe that this connection is useful also in other contexts than shortest path computations. It allows to interpret tree decompositions as multilevel graph partitions.

Our PACE 2017 submission uses the multilevel graph view. It is based upon recursive bisection respectively nested dissection. It uses FlowCutter [16] to bisect graphs.

1.1 Outline

We first define the terminology used in this paper. Afterwards, we present a high level overview of related shortest path algorithms. We then present in the next step, how multilevel partitions and tree decompositions relate. Using the multilevel partition view, we describe our PACE 2017 algorithm. Finally, we present the PACE 2017 results.

2 Definitions

In this section, we start by defining standard tree decomposition and chordal graph concepts. Afterwards, we formally define the notion of multilevel partition.

An *edge cut* C of a graph $G = (V, E)$ is a non-empty edge set. C decomposes G into the connected components of $G \setminus C$. Similarly, a *node separator* S of a graph $G = (V, E)$ is a non-empty node set. S decomposes G into the connected components of $G \setminus S$. Usually, one requires there to be at least two connected components in $G \setminus S$. However, we allow for the degenerate case of there only being one connected component. We say that a cut or separator *separates* two nodes, if they are in different components.

An undirected graph is *chordal*, if for every cycle Z with at least four nodes, there exists an edge between two nodes of Z that are not adjacent within Z . A *chordal supergraph* G' of a graph G is a supergraph of G that is chordal. *Triangulated graph* is a synonym for chordal graph, also used in the literature. An *elimination order* of an undirected graph G is an order O of the nodes of G . A supergraph G' is obtained by iteratively contracting the nodes of O , i.e., removing a node and adding a clique among its neighbors. In the shortest path literature, the term *contraction order* is used. A *perfect elimination order* of a graph G is an order where no node contraction inserts edges. A graph has a such an order, if and only if, it is chordal [13]. We therefore refer to G' as *chordal supergraph* of G .

A *tree decomposition* of a graph $G = (V, E)$ is a pair (B, T) , where B is the set of *bags* and T is the *tree backbone*. Every bag $b \in B$ is a set of nodes, i.e., $b \subseteq V$. T is a tree where the bags are the nodes, i.e., B is the set of nodes of T . A tree decomposition must fulfill three criteria to be valid:

1. Every node is in a bag, i.e., $\bigcup_{b \in B} b = V$.
2. For every edge $\{x, y\}$ of G , there must be a bag $b \in B$ such that both end points are in b , i.e., $x \in b$ and $y \in b$.
3. For every node x , the subgraph of the tree backbone T induced by all bags that contain x is a tree.

A *rooted tree decomposition* is a tree decomposition, where the tree backbone is directed towards a root bag r . A rooted tree decompositions is illustrated in Figure 3b. The *width* of a tree decomposition is the maximum size of a bag plus one. The tree width tw of a graph is the minimum width over all tree decompositions.

A *cell* c of a graph $G = (V, E)$ is a node subset $c \subseteq V$. Two cells a and b *touch*, if there is an edge with endpoints in a and b or when a and b share a node. The touching relationship is illustrated in Figure 1. The *boundary* b of c is the set of nodes adjacent to a node in c but not in c .

A *multilevel partition* P is a set of cells. Two criteria must be fulfilled for P to be valid:

1. V is a cell. We refer to V as the *toplevel cell*.
2. Touching implies nesting. If two cells a and b touch then $a \subseteq b$ or $b \subseteq a$.

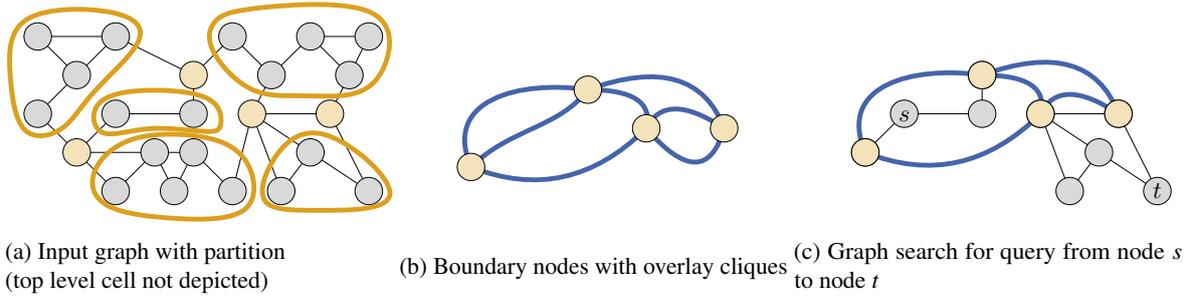


Figure 2: MLD example on bilevel partition.

p is a *parent cell* of c if $c \subset p$ and no other cell q exists with $c \subset q \subset p$. Analogous, c is a *child* of p . In a *bilevel partition*, only the toplevel cell has children. A bilevel partition is illustrated in Figure 2a and a multilevel partition in Figure 3a.

The term “multilevel partition” is often used in experimental algorithms papers [8] but rarely formally defined. It usually describes an informal algorithm design pattern. The exact details therefore significantly vary between application-focused papers. However, there is a significant difference between our formalization and many other papers. We separate cells using node separators. Many application-focused papers use edge cuts. We use separators as it enables a tighter coupling with tree decompositions.

3 Application: Shortest paths in Road graphs

Multilevel Dijkstra (MLD) and Contraction Hierarchies (CH) are two very successful shortest path acceleration techniques. In this section, we briefly outline the algorithms.

3.1 Multilevel Dijkstra

The core idea of Multilevel Dijkstra (MLD) was introduced in [18] using a bilevel partition. It was extended to multiple levels in [17]. The algorithm was refined and popularized by [10]. A parameterized analysis of the algorithm using tree width is provided by [1]. We describe the initial bilevel algorithm as it is the simplest. This technique is also often called multilevel overlay graph (MLO) or CRP which references the title of [10].

The first step of the preprocessing consists of computing a set of non-touching cells. This is illustrated in Figure 2a. In the second step, an *overlay* is computed for every cell c . Denote by G_c the subgraph induced by c and its boundary. Further, denote by H_c the overlay graph. H_c is substitution for G_c that maintains the shortest path distances among the boundary nodes. H_c must contain at least the boundary nodes. In theory, H_c can be very complex graphs. However, usually a clique among the boundary nodes is used. The weights of the clique edges are computed by running Dijkstra’s algorithm from every boundary node restricted to G_c . Figure 2b illustrates the overlays. The query consists of running a bidirectional variant of Dijkstra’s algorithm. It explores the cells of s and t and replaces all other cells with their overlays. Figure 2c illustrates the explored subgraph.

The idea of this algorithm can easily be applied recursively. The extension uses a multilevel partition as input. In [1], the MLD distance query running time with clique overlays was bounded by $O(tw^2(\log n)(\log(tw \log n)))$.

3.2 Contraction Hierarchies

The algorithm was originally described in [14]. A newer variant of the algorithm is called Customizable Contraction Hierarchies (CCH) [12]. We outline the later variant, because it is tighter coupled with tree decompositions.

The preprocessing consists of several steps. In the first step, a contraction order O is computed using nested dissection. In the second step, the nodes are contracted in this order. The inserted edges are called *shortcuts*. The graph plus the shortcuts is the CCH. In different contexts, the contraction order is called elimination order, the shortcuts are the *fill-in*, and the CCH is the chordal supergraph.

The order O is usually interpreted as ordering the nodes from bottom to top. The top node is the last node of O and contracted last. A node x is *lower* than y , if x comes before y in O . Denote by $w(x, y)$ the weight of edge $\{x, y\}$. The objective is to enforce the *lower triangle inequality*. For every triangle $\{x, y, z\}$ in the CCH, it requires that $w(x, z) + w(z, y) \geq w(x, y)$, where z is lower than x and y . The lower triangle inequality is enforced using a triangle listing algorithm. All edges already present in the input graph are assigned their input weight. Shortcuts are assigned ∞ . Triangles $\{x, y, z\}$ are enumerated ordered by the position of the lowest node z in O . If $w(x, z) + w(z, y) < w(x, y)$, then $w(x, y)$ is set to $w(x, z) + w(z, y)$.

A query for a st -path consists of a bidirectional graph search from s and t . From a node x both searches only follow edges to nodes higher than x . Once both searches are terminated, a shortest *up-down* st -path P is found. P is a path

Cell Interior	Cell Boundary	Corresponding Bag
a	b, f	a, b, f
a, b	f, c	b, f, c
n	m, o	n, o, m
n, o	m, p	o, m, p
m, o, n	f, p	f, m, p
g	f, c, p	f, g, p, c
a, b, f, g, m, n, o	p, c	f, p, c
j, k, l	r, i	j, r, i
k, l	j	k, l, j
q	p, i, r	q, r, p, i
d	c, e	d, e, c
h	c, e, p	h, e, p, c
d, e, h	c, p, i	e, i, p, c
q, r, j, k, l	p, i	p, r, i
d, e, h, i, q, r, j, k, l	p, c	p, c, i
a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r	\emptyset	p, c

Table 1: Interior, boundary, and bags of the cells of the multilevel partition of Figure 3.

$v_1 \rightsquigarrow v_i \rightsquigarrow v_k$ such that the nodes between v_1 and v_i are ascending according to O . The nodes from v_i to v_k are descending according to O . The query is correct, if there always exists an up-down path that is as long as a shortest path. To show this consider a shortest path P . If P is an up-down path, we are done. Otherwise, there exists a subpath $x \rightarrow z \rightarrow y$ such that z is lower than x and y . As z is contracted before x and y , there exists a shortcut from x to y . Further, because of the lower triangle inequality, the shortcut's weight is as long as the path $x \rightarrow z \rightarrow y$. We can thus remove z from P . Either P is now an up-down path or we apply the argument iteratively. As we eventually always end up with a shortest up-down path, the query is correct.

The maximum cliques in a CCH/chordal supergraph correspond to the tree decomposition. It was shown in [4] that the number of nodes visited by one side is bounded by $O(\text{tw} \log n)$. As the explored subgraphs can be dense, the distance query runs in $O(\text{tw}^2 \log^2 n)$ time.

4 Connection between Tree Decompositions and Multilevel Partitions

In this section, we prove a one-to-one correspondence between multilevel partitions and rooted tree decompositions. A consequence is that both concepts are just two different views onto the same object.

We first describe the algorithm to translate a multilevel partition into a rooted tree decomposition. In the next step, we describe the algorithm to perform the inverse operation. Finally, we show that the outputs of both algorithms are valid and that chaining both algorithms is the identity function.

4.1 From Multilevel Partition to Rooted Tree Decomposition.

For every cell c in the multilevel partition, we construct a bag b in a tree decomposition. b is the union of the boundary and interior nodes of c minus the interior nodes of all children. The parent-child relation between cells induces a tree on the bags. This is the tree backbone. The top level cell is the root of the rooted tree backbone. The so obtained tree decomposition can be degenerate, i.e., it is possible that bags exist that are subsets of other bags.

4.2 From Tree Decomposition to Multilevel Partition

A tree decomposition T does not uniquely define a multilevel partition P because the tree backbone does not have a root. The transformation from T must therefore start by picking a root bag r . With respect to r , we can construct for every bag b except the root a cell as follows: Denote by p the parent of b , i.e., the first node on the unique path from b to r . We set the boundary of cell c to $b \cap p$. The interior of c is set to the union of all direct or indirect children bags of c minus c 's boundary. We additionally construct a cell for the root bag. This cell's boundary is empty and its interior is the whole graph.

4.3 Example

Figure 3a depicts a multilevel partition with 16 cells. Every cell (except the top level cell) is depicted as closed curve. The color of a curve indicates the recursion depth. Orange indicates depth 1, blue depth 2, green has depth 3, and red indicates

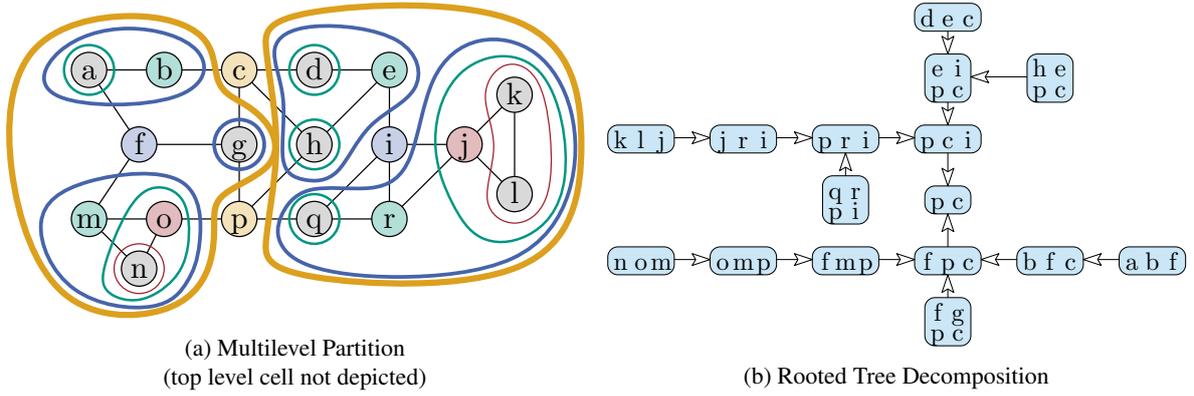


Figure 3: Multilevel Partition and corresponding rooted tree decomposition.

depth 4. Grey nodes are not part of any separator. The color of the remaining nodes indicates the depth of the separator that they are part of. Table 1 enumerates all cells in the multilevel partition and the derived bags. Together with the parent-child relation of the cells, we obtain the rooted tree decomposition depicted in Figure 3b.

There are cells, such as the cell with interior $\{a, b\}$, which is “divided” along the separator $\{b\}$ into one part, namely the cell with interior $\{a\}$. Sufficiently large cells are usually divided into more than one part. However, for tiny cells, it often occurs that such awkward separators are used.

Figure 4 depicts the nodes in a mostly German road graph at their geographical positions. Every dot is a node. The nodes in the largest bag of a tree decomposition are highlighted. This bag has 369 nodes. The Figure depicts an area surrounded by orange dots. This area is divided into two parts along separator consisting of further orange dots.

The nodes in the area are the nodes in the cell. The orange separator nodes are also in the cell. However, the orange nodes that surrounded the area are not inside the cell. They are in its boundary.

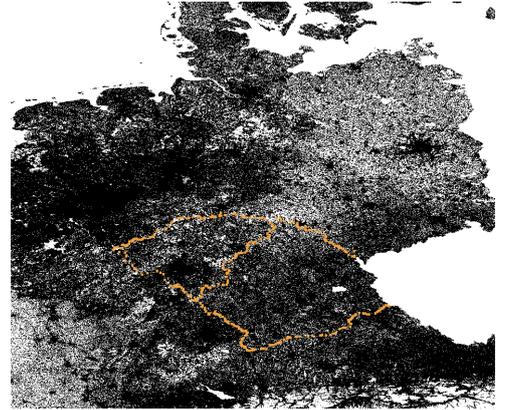


Figure 4: Largest Bag in Road Graph Tree Decomposition.

4.4 Correctness

Lemma 1. *The constructed tree decomposition is valid.*

Proof. We need to show that the three conditions laid out in the tree decomposition definition are fulfilled. We need to show that every node is in a bag. To prove this, we observe that every node is interior to the top level cell. A node v interior to a cell c is either in v ’s bag or interior to a child of c . As every cell has a finite number of descendants, we cannot build infinite chains of nested cells. We have thus proven that every node is in a bag.

Further, we need to show that for every edge $\{x, y\}$ there exists a bag b such that x and y are part of b . As touching cells are ordered by inclusion and as there are only finitely many cells, we know that there exists a smallest cell c_x that has x in its interior. x is in the bag of c_x because x is in c_x but not in the interior of a child of c_x . Let c_y be the analogous smallest cell for y . If $c_x = c_y$, then c_x is the required b . Otherwise, we observe that the existence of $\{x, y\}$ implies that c_x and c_y touch each other. They are therefore ordered by inclusion. Assume without loss of generality that $c_x \subseteq c_y$. The existence of $\{x, y\}$ implies that y is on the boundary of c_x and therefore in the bag of c_x . c_x is therefore the required bag b .

Finally, we need to show that for every node x the set of bags that include x forms a subtree of the backbone. Consider again the smallest cell c_x that contains x . All cells that contain x are ancestors of c_x . As x is in c_x , x cannot be in the bag of any ancestor of c_x . We therefore know that c_x is the only cell that has x in its interior and bag. Pick another cell d whose bag contains x . As $d \neq c_x$, we know that x is on d ’s boundary. Denote by p the parent cell of d . x is in p or on the boundary of p . We can thus conclude that x is in the bag of p . From d we can iteratively follow the parent relation. As the parent-child relation is acyclic, we eventually arrive at c_x . As for every d the corresponding path ends at c_x , we have proven that the set of bags that include x forms a subtree of the backbone.

As we have proven all three properties, we have proven that the constructed tree decomposition is valid. \square

Lemma 2. *The constructed multilevel partition is valid.*

Proof. We need to show that touching cells are ordered by inclusion. Cells touch if one of two conditions is fulfilled:

- They share a node.

- There exists an edge with endpoints in both cells.

We show the required ordering property independently for both cases.

Consider some arbitrary node x and denote by T_x the subtree of the backbone induced by x . T_x contains a unique bag b_x that is closest to the root. The bags in the tree decomposition fall into three categories:

1. They are in T_x but are different from b_x .
2. They lie on the unique path from b_x to the root.
3. They are neither in T_x nor on the path.

We show that only the cells corresponding to the bags of category 2 contain x . The cells along a path are trivially ordered by inclusion.

Let q be a bag from category 1. As it is different from b_x , it cannot be the root. Therefore, there exists a parent bag p of q . By construction x is also contained in p . We have that $x \in q \cap p$. x is in the boundary and not in the interior of the cell corresponding to q .

Now let q be a bag from category 3. The corresponding cell is constructed by forming the union of bags that do not contain x . The union thus also does not contain x .

Finally, let q denote a bag on the path. The constructed union contains all bags of T_x and thus also contains x . It remains to show that x is not on the boundary of the corresponding cell. This follows from the fact that b_x is the only bag that contains x . As b_x is the bag the farthest away from the root, no parent bag of a bag on the path is b_x . x is thus not part of any boundary.

This completes the first part of the proof. Next consider the case where the cells corresponding to two bags b_x and b_y touch because there exists an edge $\{x, y\}$ between them. By convention, we set $x \in b_x$ and $y \in b_y$. We know from the second property of the tree decomposition definition that there exists a bag b_{xy} that contains x and y .

We know that the tree backbone must contain the following four paths:

- There is a path D_x from b_{xy} to b_x along T_x because trees are connected.
- Using an analogous argument, we know that there exists a path D_y from b_{xy} to b_y along T_y .
- We can follow the parent relation from b_x to the root and obtain a path U_x .
- Analogously, there exists a path U_y from b_y to the root.

By concatenating all four paths U_x , U_y , D_y , and D_x , we obtain a cycle in the tree backbone. As the tree backbone is a tree, we conclude that the cycle must be degenerate. The set of the root r , b_x , b_y , and b_{xy} can therefore only contain at most two elements. We conclude that one of the following conditions must hold as otherwise the set would contain three or more elements: $b_x = b_y$, $b_x = r$, or $b_y = r$. In the first case, the cells are equal and thus ordered. In the second and third cases, one of the cells is the root cell and thus by definition a superset of the other cell.

This completes the second and last part of the proof. We have proven that the constructed multilevel partition is valid. \square

Lemma 3. *Applying both algorithms after another is the identity.*

Proof. As both algorithms essentially just copy the tree backbone, it is clear that it is not modified. It remains to show that the contents of the bags and cells remains unchanged.

Pick a cell c from the multilevel partition and denote by b the corresponding bag. b is by construction a subset of the union c 's boundary and its interior. The missing nodes must be in the interior of a child cell of c and therefore in the bag of a descendant of b . The union over the bags of the subtree rooted at b is thus equal to the union of c 's boundary and interior. Removing the boundary from this union yields the interior, as the interior and the boundary are disjoint. The interior of c is thus left unchanged after applying both algorithms. Applying both algorithms is therefore the identity. \square

5 PACE 2017 FlowCutter Submission

Our algorithm is based on the nested dissection paradigm [15]. In the graph partitioning literature, this approach is often called recursive bisection. Compared to the original algorithms, ours does not recurse until all graph parts are smaller than a given threshold. Instead, it aborts early, when it detects that the decomposition width can no longer be improved.

Internally, our algorithm uses the multilevel partition representation described in the previous section. It translates the data to a tree decomposition when formatting the result. For every cell c , it stores two sets B_c and I_c . Both sets contain nodes of the input graph. B_c contains the nodes on the boundary of c . I_c contains the interior nodes of c that are not in a child cell. The bag corresponding to c is $B_c \cup I_c$. We refer to $|B_c| + |I_c|$ as the bag size of c . The maximum bag size over all cells in a multilevel partition is the corresponding tree decomposition width minus one.

Our algorithm maintains a multilevel partition. The cells are organized in two sets \mathcal{F} and \mathcal{O} . \mathcal{F} is the set of final cells. \mathcal{O} is the set of open cells. \mathcal{F} contains the cells that can no longer be modified. The cells in \mathcal{O} can be modified using a cell split operation. Splitting c yields one final cell and several possibly zero open cells. No new cell has a bag size larger than the one of c .

The set \mathcal{O} is organized as maximum priority queue using the bag size as key. Our algorithm further maintains the maximum bag size over all final cells. Initially, the set \mathcal{F} is empty. We define the maximum bag size over an empty \mathcal{F} as 0. Initially, \mathcal{O} contains a single toplevel cell c that contains all nodes. This means that $B_c = \emptyset$ and $I_c = V$.

Our algorithm iteratively splits cells. It picks the cell c from \mathcal{O} that maximizes the bag size. If c 's bag size is smaller or equal to the maximum bag size over all final cells, our algorithm terminates. Otherwise, it removes c from \mathcal{O} and splits c . This yields a final cell c_f and several possible zero open cells $c_1 \dots c_k$. Our algorithm adds c_f into \mathcal{F} and updates the stored maximum bag size. The cells $c_1 \dots c_k$ are added into \mathcal{O} .

5.1 Splitting a Cell

In this subsection, we describe how a cell c is split. Consider the subgraph G_c induced by I_c . Our algorithm computes a balanced separator S_c using FlowCutter. The resulting cells are derived from this separator. The final cell c_f is similar to c except that nodes are removed from I_c . Formally, $B_{c_f} = B_c$ and $I_{c_f} = S_c$. The resulting open cells correspond to the connected components of $G_c \setminus S_c$. For every component there is a cell c_c . Our algorithm sets I_{c_c} to the nodes in the corresponding component. B_{c_c} is a subset of $B_c \cup S_c$. It only contains the nodes adjacent to a node in I_{c_c} .

To test the adjacency, we maintain an array that maps every node onto a bit. Initially, we set each bit to false. For all nodes in I_{c_c} , we set the bit to true. Next we iterate over all nodes x in $B_c \cup S_c$. We iterate over the neighbors of x in the input graph. If the bit of one neighbor is set, we add x to B_{c_c} . Finally, we reset the bit of all nodes in I_{c_c} to false.

5.2 FlowCutter

FlowCutter is a graph bisection algorithm described in [16]. In this subsection, we present the high level ideas. For all details, we refer to [16]. The base algorithm computes balanced edge cuts. It can be extended to compute balanced node separators. Our algorithm interprets the graphs as symmetric unit flow network. It repeatedly solves multi-source multi-target maximum flow problems. Initially, there is only one source and one target node. They are picked uniformly at random. Our algorithm starts by computing a maximum flow. From this flow, it derives the least balanced minimum cut C . If C is sufficiently balanced, it terminates. Otherwise, it iteratively improves the cut in rounds. New source or target nodes are added in each round. Suppose that C is unbalanced. The algorithm thus continues with an additional round. Further, assume without loss of generality that the source's side is smaller than the target's side. Our algorithm marks all nodes on the source's side as additional sources. Additionally, it marks one node on the target's side as source node. This node is called *piercing node*. It is an endpoint of a cut edge. Choosing the piercing node is difficult. If there is a choice that does not increase the cut size, our algorithm picks it. Otherwise, it employs a heuristic based on the distances from the original source and target nodes. After adding the additional sources, C no longer separates all source nodes. In the next round a different cut is found. Usually, we run several FlowCutter instances in parallel with different initial source and target pairs. We refer to the number of instances as the number of *cutters*. In each round FlowCutter finds a new cut. During its execution our algorithm thus does not only compute a single balanced cut. Instead, it computes a set of cuts that heuristically optimize balance and cut size in the Pareto-sense. This allows us to optimize complex criteria heuristically. For example, we can pick a cut that minimizes the ratio between the cut size and the smaller side size, i.e., with minimum expansion. From our experience, minimizing the expansion subject to a bounded balance, heuristically yields good results. In flow problems, node capacities can usually be reduced to edge capacities [2]. Using such a reduction, FlowCutter can be used to compute node separators. Alternatively, node separators can be derived from edge cuts by picking for every cut edge an endpoint. The former can achieve smaller separators but is slower as the graph needs to be expanded.

5.3 Submission Details

Our submission runs FlowCutter in an endless loop until the maximum execution time of 30 min is reached. It uses node capacities and minimizes the expansion subject to a bounded balance. In each iteration, our submission varies some parameters such as the number of cutters, the minimum required balance, or the random seed. For small instances, we compute heuristic elimination orders before running FlowCutter. From these orders, we derive tree decompositions. We compute an order that greedily picks a minimum degree node. Another order greedily picks a minimum fill-in node. We compute these orders because the derived tree decompositions are sometimes slightly smaller than those computed by FlowCutter. However, on large instances computing them using our implementation is not possible within the maximum execution time of 30 min. On the large instances, we configure FlowCutter to compute edge cuts and minimize the cut size subject to a bounded balance. From the edge cuts, we derive node separators by picking an endpoint of every edge. This is necessary to assure that at least one iteration finishes within the allowed execution time. Our implementation finds a new tree decomposition each time a cell is split. Formatting the textual output corresponding to the new tree decomposition can be slower than performing the split operation. To avoid these costs, we produce a new textual output only every 30 s.

Competitor	Approx. Ratio		Diff. to best Sol. [%]			
	Avg.	Max.	= 0	≤ 1	≤ 2	< ∞
FlowCutter	1.08	1.55	42	50	56	100
Solver by Abseher, Musliu, Woltran	1.11	2.37	25	37	46	95
Solver by Bannach, Berndt, Ehlers	1.19	3.90	21	29	32	93
Solver 1 by Jégou, Kanso, Terrioux	1.30	2.43	9	13	17	81
Solver by Tamaki, Ohtsuka, Sato, Makii	1.30	9.61	51	52	52	92
Solver by Larisch, Salfelder	1.56	3.74	6	8	13	54
Solver 2 by Jégou, Kanso, Terrioux	4.68	139.70	10	14	18	71

Table 2: Performance Comparison of all PACE 2017 Track A2 competitors.

6 Evaluation

To evaluate our algorithm, we submitted it the PACE 2017 challenge. We entered track A2. The objective is to compute a tree decomposition of small size within 30 min. All competitors were evaluated on the same set of test graphs. The algorithms that find a smallest decomposition win the instance. Entries were ranked using the Schulze method. This usually implies that the competitor that wins the most instances wins the competition. Our submission won the second place. The submission by Tamaki et al. won the competition. Abseher et al. is ranked third. The difference between the first two places is small. In the following, we look at the results in depth.

Denote by t^* the best solution found for an instance and by t the solution found by a competitor. A weakness of the PACE evaluation method is that it does not consider how large $t - t^*$ is. $t - t^* = 1$ is weighted in the same way as $t - t^* = 1000$. Only the relative ranking among competitors matters. To investigate the impact of this effect, we present in Table 2 alternative rankings. We report relative approximation ratios. Denote by t^* the best solution found for an instance and by t the solution found by a competitor. We report the average and maximum t/t^* over all non-trivially¹ solved instances. With respect to both criteria, our submission clearly wins. Interestingly, the competition winner is only ranked fifth with respect to the average t/t^* . Additionally, we report how often $t - t^*$ is below a threshold. For instances that were not or only trivially solved, we set $t = \infty$. “= 0” is the number of instances where a competitor produced a best solution. “≤ X” indicates how often it was off by X. “< ∞” is the number of non-trivially solved instances.

The algorithm by Tamaki et al. clearly wins with respect to “= 0”. The gap with our algorithm is only 2% for “≤ 1”. For “≤ 2” our algorithm wins. Further, our algorithm is the only competitor that finds non-trivial solutions on all instances. The algorithm by Tamaki et al. finds on many small instances solutions that are one or two nodes smaller than our algorithm. For large instances, their algorithm often does not find a solution.

7 Conclusion

We conclude that the winning algorithm by Tamaki et al. is good, if the instances are small and achieving the smallest width is of utmost importance. If the instances are large or finding nearly the best solutions is good enough, our algorithm is better. Further, we illustrated that viewing tree decompositions as multilevel partitions is a useful algorithm design tool.

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¹A non-trivial solution must have width at most $|V| - 5$. Criterion was defined by PACE organizers.

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